SUBJECT: MATHEMATICS

PAPER: *NUMERICAL ANALYSIS*

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NUMERICAL ANALYSIS (M. Sc./ M.A.)

Unit I

Errors in computation: Floating point representation of numbers, Significant digits, Rounding and chopping a number and error due to these absolute and relative errors, Computation of errors using differentials, Errors in evaluation of some standard functions, Truncation error.

Linear equations: Gauss elimination method, LU Decomposition method, Gauss–Jordan method, Tridiagonal system, Inversion of matrix, Gauss-Jacobi, Gauss-Seidal iterative methods and their convergence

Unit II

Non-linear equations: Iterative method, Secant method, Rate of convergence of Regula-Falsi method, Newton-Raphson method, Convergence of Newton-Raphson method for simple and multiple roots, Birge-Vieta method, Bairstow's method and Graffe's root squaring method for polynomial equations.

Unit III

Numerical differentiation: Differentiation methods based on Newton's forward and backward formulae, Differentiation by central difference formula.

Numerical integration: Methodology of numerical integration, Rectangular rule, Trapezoidal rule, Simpson's $1/3^{rd}$ and $3/8^{th}$ rules, Romberg Integration, Gauss-Legendre quadrature formula.

Unit IV

Algebraic Eigen values and Eigen vectors: Power method, Jacobi's method, Given's method, Householder's method, Approximation: Least square polynomial approximation, polynomial approximation using orthogonal polynomials, Approximation with algebraic and trigonometric functions.

Unit V

Ordinary differential equations: Initial and boundary value problems, Solutions of Initial Value Problems, Single and multistep methods, Picard's method, Taylor series method,Euler's and Modified Euler's methods, Runge-Kutta second order and fourth order methods, Milne's method, Adams-Bashforth method.

RECOMMENDED BOOKS

1. **Radhey S. Gupta**,Elements of Numerical Analysis, Macmillan India Ltd. New Delhi.
2. **M.K.Jain, S.R.K.Iyengar, R.K.Jain,** Numerical Methods for Scientific and Engineering Computations, New Age International (P) Ltd. New Delhi .
3. E.V. Krishnamurthy and S.K. Sen, Computer Based Numerical Analysis, PHI.
4. B. Bradie : A Friendly Introduction to Numerical Analysis, PEARSON.

(Y.Vimla) (S.P. Gupta) (R.C. Mittal) (V.P. Kaushik) (R.C. Dimri)

16-11-13

(Mridul Kumar Gupta) (Jaimala) (Raj Pal Singh) (Harikishen)

UNIT II: "Algebraic and Transcendental Equations"

An expression of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

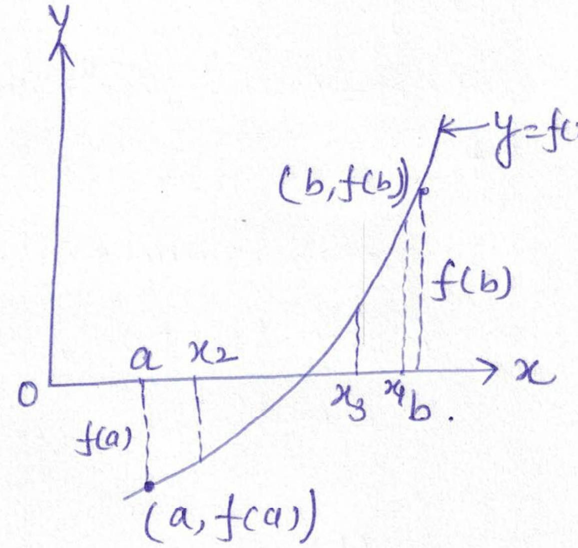
where all a's are constant provided $a_0 \neq 0$ and n is a positive integer, called a polynomial of degree n in x. The polynomial equation $f(x)=0$ is called an algebraic equation of degree n. But when $f(x)$ is an expression involving some other functions such as trigonometric, logarithmic, exponential etc then $f(x)=0$ is called a transcendental equation.

Now, we will discuss some numerical methods for the solution of $f(x)=0$, where $f(x)$ is algebraic or transcendental or both. The value of x for which an equation $f(x)=0$ is satisfied, is called its root. Geometrically, a root of an equation $f(x)=0$ is that value of x where the graph of $y=f(x)$ cuts the x-axis.

The process of finding the roots of an equation is known as the solution of that equation. Higher degree equations or transcen-dental equations can be solved by approximation methods such as Bisection method, Secant method, Regula-falsi method and Newton's method etc.

BISECTION METHOD (OR BOLZANO METHOD)

This method is based on the theorem, which states that "if a function $f(x)$ is continuous in the closed interval $[a, b]$ and $f(a)$, $f(b)$ are of opposite signs, then there must exist at least one root (real) between $x = a$ and $x = b$.



Let the function $f(x)$ be continuous between a and b. Without loss of generality, let $f(a)$ be (−)ve and $f(b)$ be (+)ve. Then the first approximation to the root is

$$x_1 = \frac{a+b}{2}$$

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$ otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is (+)ve or (−)ve. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

In the adjoining figure, $f(x_1)$ is (+)ve so that the root lies between a and x_1. Then second approximation to the root is

$$x_2 = \frac{1}{2}(a + x_1).$$

If $f(x_2)$ is $(-)$ve ; the root lies between x_1 and x_2. Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2)$ and so on.

Order of convergence of iterative method

Convergence of an iterative method is defind as the order at which the error between successive approximations to the root decreases.

An iterative method is said to be k^{th} order convergent if k is the largest positive real number such that

$$\lim_{n \to \infty} \left| \frac{e_{n+1}}{e_n^k} \right| \leq A \qquad (1)$$

where A is a non-zero finite number called asymptotic error constant and it depends on the derivative of $f(x)$ at an approximate value of x. e_n and e_{n+1} are the errors in n^{th} and $(n+1)^{th}$ approximations to the root respectively.

Order of convergence of Bisection Method :-

In Bisection method, the original interval is divided into half interval in each iteration. If we take mid points of successive intervals to be the approximations of the root, one half of the current interval is the upper bound to the error.

In Bisection method,

$$e_{n+1} = \frac{1}{2} e_n$$

or, $$\frac{e_{n+1}}{e_n} = \frac{1}{2} \quad \text{——— (1)}$$

where e_n and e_{n+1} are the errors in the n^{th} and $(n+1)^{th}$ iterations respectively.

comparing eq^n (1) with

$$\lim_{n \to \infty} \left| \frac{e_{n+1}}{e_n} \right| \leq A$$

we get $k = 1$ and $A = \frac{1}{2}$

Thus the Bisection method is I order convergent or linearly convergent.

Assignment :- Prove that Bisection method is always convergent.

Question :- Find the root of the equation
$$f(x) = x^3 - 4x - 9 = 0,$$
using the bisection method in four iterations.

Solution :- Given $f(x) = x^3 - 4x - 9 = 0$

then $f(1) = 1^3 - 4\times1 - 9 = -12$

$f(2) = 2^3 - 4\times2 - 9 = -9$

$f(3) = 3^3 - 4\times3 - 9 = 6$

$\therefore$ $f(2)$ is negative & $f(3)$ is positive, so a root lies between 2 and 3.

$\therefore$ first approximation to the root is
$$x_1 = \frac{2+3}{2} = 2.5$$
$$f(x_1) = f(2.5) = (2.5)^3 - 4(2.5) - 9$$
$$= 15.625 - 10 - 9$$
$$= -3.375$$

Since $f(2.5)$ is $(-)$ve and $f(3)$ is $(+)$ve therefore the root lies between 2.5 and 3. Hence the second approximation to the root is
$$x_2 = \frac{2.5+3}{2} = 2.75$$

now, $f(2.75) = (2.75)^3 - 4(2.75) - 9$
$$= 20.797 - 20 = 0.797$$

Since $f(2.75)$ is $(+)$ve and $f(2.5)$ is $(-)$ve. Thus the root lies between 2.5 and 2.75. Hence the third approximation to the root is given by

$$x_3 = \frac{2.5 + 2.75}{2} = 2.625$$

Now $f(2.625) = (2.625)^3 - 4(2.625) - 9$

$$= 18.088 - 19.5 = -1.412$$

Since $f(2.625)$ is $(-)$ve and $f(2.75)$ is $(+)$ve therefore the root lies between 2.625 and 2.75. Hence, the fourth root (approximation to the root) is

$$x_4 = \frac{2.625 + 2.75}{2}$$

$$= 2.6875$$

Thus, after the four iteration the root is 2.6875 approximately.

Assignment :-

Q1. using Bisection method determine a real root of the equation $f(x) = 8x^3 - 2x - 1 = 0$.

Q2. Find a real root of $x^3 - x = 1$ lying between 1 and 2 by Bisection method. Compute five iteration.

Q3. Find a real root, correct to three decimal places for $x^3 - x - 4 = 0$ using Bisection method.

Q4. using Bisection method, determine a real root of the equation $f(x) = 8x^3 - 2x - 1 = 0$ correct to two decimal place..

The method of Iteration

In order to find the roots of the equation $f(x)=0$ by successive approximations, we write it in the form $x=\phi(x)$.

The roots of $f(x)=0$ are the same as the points of intersection of the straight line $y=x$ and the curve representing

$$y=\phi(x)$$

Let $x=x_0$ be an initial approximation of the desired root α then first approximation x_1 is given by

$$x_1=\phi(x_0)$$

Now, treating x_1 as the initial value, the second approximation is $x_2=\phi(x_1)$.

Proceeding in this way, the n^{th} approximation is given by

$$x_n=\phi(x_{n-1})$$

Remark :- The iternation method is applicable if $|\phi'(x)|<1$ for $x\in(a,b)$.

Example :- Find a real root of the equation

$$\cos x = 3x - 1$$

Correct to 3 decimal places using iteration method.

Solution :- we have $f(x) = \cos x - 3x + 1 = 0$.

Now $f(0) = \cos 0 - 3 \times 0 + 1$

$= 2$

$f(\pi/2) = -\frac{3\pi}{2} + 1 = -ve$

$\therefore$ A root lies between 0 and $\pi/2$

Rewritting the given eq^n as

$$x = \frac{1}{3}(\cos x + 1) = \phi(x)$$

$$\therefore \quad \phi'(x) = -\frac{\sin x}{3}.$$

and $|\phi'(x)| = \frac{1}{3}|\sin x| < 1$ in $(0, \pi/2)$

hence iteration method can be applied and we start with $x_0 = 0$ then the successive approximations are

$$x_1 = \phi(x_0) = \frac{1}{3}(\cos 0 + 1) = 0.6667$$

$$x_2 = \phi(x_1) = \frac{1}{3}[\cos(0.6667) + 1] = 0.5953$$

$$x_3 = \phi(x_2) = \frac{1}{3}[\cos(0.5953) + 1] = 0.6093$$

$$x_4 = \phi(x_3) = \frac{1}{3}[\cos(0.6093) + 1] = 0.6067$$

$$x_5 = \phi(x_4) = \frac{1}{3}[\cos(0.6067) + 1] = 0.6072$$

$$x_6 = \phi(x_5) = \frac{1}{3}[\cos(0.6072) + 1] = 0.6071$$

$\because$ x_5 and x_6 are almost same, the root is 0.607 correct to three decimal places.

The method of Iteration for System of Non-linear equations :-

let the equation be $f(x,y)=0$, $g(x,y)=0$ whose real roots are required within a specified accuracy

we assume $x = F(x,y)$ and $y = G(x,y)$

where functions F and G satisfy conditions

$$\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial F}{\partial y}\right| < 1 \quad \text{and} \quad \left|\frac{\partial G}{\partial x}\right| + \left|\frac{\partial G}{\partial y}\right| < 1$$

in nbd of root.

let (x_0, y_0) be the initial approximation to a root (α, β) of the system. we construct then successive approximations as

$$x_1 = F(x_0, y_0) \quad , \quad y_1 = G(x_0, y_0)$$
$$x_2 = F(x_1, y_1) \quad , \quad y_2 = G(x_1, y_1)$$
$$x_3 = F(x_2, y_2) \quad , \quad y_3 = G(x_2, y_2)$$

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$$x_{n+1} = F(x_n, y_n) \quad , \quad y_{n+1} = G(x_n, y_n)$$

If iteration process converges then, we get

$$\alpha = F(\alpha, \beta)$$
$$\beta = G(\alpha, \beta)$$ in the limit

Thus α, β are the roots of the system.

Question :- Find a real root of the equations by iteration method.

$$x = 0.2x^2 + 0.8 \quad , \quad y = 0.3xy^2 + 0.7$$

Solution :- we have $F(x,y) = 0.2x^2 + 0.8$

$$G(x,y) = 0.3\,xy^2 + 0.7$$

$$\frac{\partial F}{\partial x} = 0.4x \qquad \frac{\partial G}{\partial x} = 0.3y^2$$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial G}{\partial y} = 0.6xy$$

It is easy to see that $x=1$ and $y=1$ are the roots of the system. Choosing $x_0 = 1/2$, $y_0 = 1/2$, we find that

$$\left|\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} + \left|\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} = 0.2 < 1$$

$$\left|\frac{\partial G}{\partial x}\right|_{(x_0,y_0)} + \left|\frac{\partial G}{\partial y}\right|_{(x_0,y_0)} = 0.225 < 1$$

$\therefore$ Conditions are satisfied. Hence

$$x_1 = F(x_0, y_0) = \frac{0.2}{4} + 0.8 = 0.85$$

$$y_1 = G(x_0, y_0) = \frac{0.3}{8} + 0.7 = 0.74$$

For II approximation, we have

$$x_2 = F(x_1, y_1) = 0.2\,(0.85)^2 + 0.8 = 0.9445$$

$$y_2 = G(x_1, y_1) = 0.3\,(0.85) \times (0.74)^2 + 0.7$$
$$= 0.81$$

converges to root $(1,1)$ is obvious.